Attractor for the mean-field equations of the hysteretic dynamics of a quantum spin model: Analytical solution

Subir K. Sarkar* and Debashish Bose†
School of Physical Sciences, Jawaharlal Nehru University, New Delhi 110 067, India
(Received 20 June 1996)

Recently Banerjee, Dattagupta, and Sen derived the mean-field equations for the three components of the magnetization in the context of a quantum spin model in a rotating external magnetic field [Phys. Rev. E **52**, 1436 (1995)]. We provide here the analytical solution for the attractor of this dynamics and prove that there is no hysteresis loss on the attractor. [S1063-651X(97)10102-7]

PACS number(s): 05.70.Ln, 75.60.−d, 75.10.Jm

Over the past few years there has been considerable progress in the understanding of the phenomenon of hysteresis in model systems with both large and small number of degrees of freedom [1–14]. In some of these systems, hysteresis is purely dynamical in origin, whereas in others irreversibility is caused by interaction with a heat bath. While some of these systems are purely classical in nature, others require quantum mechanical description. When many degrees of freedom are present, interaction among these variables leads to cooperative effects and as some external parameters are varied, phase transition may take place. One such system in which both cooperative and quantum effects are present was recently proposed by Banerjee, Dattagupta, and Sen [15]. They investigated the phenomenon of hysteresis in an Ising system in an external field rotating in the transverse plane. Using a mean-field approximation, they derived the dynamics for the expectation values for the three components of magnetization by using a microscopic system-plus-reservoir approach. These equations are nonlinear in nature and in Ref. [15] they were solved numerically to find the asymptotic periodic attractor. What we do in this paper is to provide an analytical solution for this asymptotic periodic attractor. In the process, we prove that there is no dissipation due to hysteresis, if the mean-field dynamics is indeed governed by the equations derived by the authors [15].

The mean-field Hamiltonian for the system-plus-reservoir that is being studied here is given by

$$H_0 = H_s + V + H_B . \tag{1}$$

Here H_s is the mean-field system Hamiltonian in the rotating transverse field and is expressed as

$$H_s = -h\sigma_z - \Gamma_x(t)\sigma_x - \Gamma_y(t)\sigma_y \tag{2}$$

with $\Gamma_x(t) = \Gamma\cos 2\omega t$ and $\Gamma_y(t) = \Gamma\sin 2\omega t$. They represent the two components of the magnetic field of strength Γ rotating in the x-y plane with frequency 2ω. σ_x, σ_y, and σ_z are the three components of the spin operator and h is the self-consistent mean-field operating in the z direction and is ultimately taken to be proportional to the average magnetization m_z in the z direction. V represents the coupling between the system and the reservoir and is invariant with respect to rotation in the x-y plane. Finally H_B is the bath Hamiltonian.

Now, define $m_x(t)$, $m_y(t)$, and $m_z(t)$ to be the expectation values of σ_x, σ_y, and σ_z, respectively, with respect to the relevant density matrix $\rho(t)$ at time t. The dynamics for $m_x(t)$, $m_y(t)$, and $m_z(t)$ as derived in Ref. [15] is finally given, for arbitrarily large values of the strength Γ of the transverse field (so that the effect of large quantum fluctuations are taken into account), by the following set of equations:

$$\begin{aligned}\frac{dm_x}{dt} = {}& m_x\left[-\lambda - \lambda\,\frac{\Gamma^2}{h_0^2}\cos^2 2\omega t\right] \\ &+ m_y\left[2h - \lambda\,\frac{\Gamma^2}{h_0^2}\sin 2\omega t\,\cos 2\omega t\right] \\ &+ m_z\left[-2\Gamma\,\sin 2\omega t - \lambda\,\frac{(h+\omega)\Gamma}{h_0^2}\cos 2\omega t\right] \\ &+ 2\lambda\,\frac{\Gamma}{h_0}\cos 2\omega t\,\tanh\beta h_0 ,\end{aligned} \tag{3}$$

$$\begin{aligned}\frac{dm_y}{dt} = {}& m_y\left[-\lambda - \lambda\,\frac{\Gamma^2}{h_0^2}\sin^2 2\omega t\right] \\ &+ m_x\left[-2h - \lambda\,\frac{\Gamma^2}{h_0^2}\sin 2\omega t\,\cos 2\omega t\right] \\ &+ m_z\left[2\Gamma\,\cos 2\omega t - \lambda\,\frac{(h+\omega)\Gamma}{h_0^2}\sin 2\omega t\right] \\ &+ 2\lambda\,\frac{\Gamma}{h_0}\sin 2\omega t\,\tanh\beta h_0 ,\end{aligned} \tag{4}$$

and

*Electronic address: sarkar@jnuniv.ernet.in
†Electronic address: sps@jnuniv.ernet.in

$$\frac{dm_z}{dt}=m_x\left[2\Gamma\ \sin 2\omega t-\lambda\ \frac{(h+\omega)\Gamma}{h_0^2}\cos 2\omega t\right]$$
$$+m_y\left[-2\Gamma\ \cos 2\omega t-\lambda\ \frac{(h+\omega)\Gamma}{h_0^2}\sin 2\omega t\right]$$
$$+m_z\left[-2\lambda+\lambda\ \frac{\Gamma^2}{h_0^2}\right]+2\lambda\ \frac{h+\omega}{h_0}\tanh\beta h_0. \qquad (5)$$

Here $h_0=\sqrt{(h+\omega)^2+\Gamma^2}$, λ is a phenomenological relaxation rate characterizing the heat bath and $\beta=T^{-1}$, T being the temperature. Once the dynamics evolves into the asymptotic periodic attractor, hysteresis loss per cycle in the dynamics is given by

$$A=\int_0^{\overline{T}}[m_x d\Gamma_x(t)+m_y d\Gamma_y(t)] \qquad (6)$$

with $\overline{T}=\pi/\omega$.

We will now find out the solution for the asymptotic periodic attractor of the set of coupled ordinary differential equations represented by Eqs. (3), (4), and (5) analytically. To do this, the first point to be noted is that the Hamiltonian without the external rotating field is symmetric with respect to rotation in the x-y plane and the external field is of constant magnitude while rotating in the x-y plane with a constant angular velocity 2ω. In order to satisfy invariance with respect to both rotation and time translation that is inherent in the dynamics, the only possible solution for the periodic attractor can be one in which the magnetization vector $(m_x(t),m_y(t),m_z(t))$ also has a constant magnitude and its projection in the x-y plane maintains a constant angular relationship with respect to the instantaneous vector $(\Gamma_x(t),\Gamma_y(t))$ representing the external field and the z component of magnetization is constant. Thus

$$m_x=\alpha\ \cos(2\omega t+\phi), \qquad (7)$$

$$m_y=\alpha\ \sin(2\omega t+\phi), \qquad (8)$$

and

$$m_x=m_0. \qquad (9)$$

Substituting these in Eq. (5), one gets

$$m_0\left(-2\lambda+\lambda\ \frac{\Gamma^2}{h_0^2}\right)+2\lambda\ \frac{h+\omega}{h_0}\tanh\beta h_0-2\Gamma\alpha\ \sin\phi$$
$$-\lambda\alpha\ \frac{(h+\omega)\Gamma}{h_0^2}\cos\phi=0. \qquad (10)$$

Similarly, substituting for m_x, m_y, and m_z from Eqs. (7), (8), and (9) in Eq. (3) and equating the coefficients of $\sin 2\omega t$ and $\cos 2\omega t$ on both sides, one obtains the following two equations:

$$2\alpha\omega\ \cos\phi=2\Gamma m_0-\lambda\alpha\ \sin\phi-2\alpha h\ \cos\phi \qquad (11)$$

and

$$2\alpha\omega\ \sin\phi=\lambda\ \frac{(h+\omega)\Gamma}{h_0^2}m_0-2\lambda\ \frac{\Gamma}{h_0}\tanh\beta h_0-2\alpha h\ \sin\phi$$
$$+\left(\alpha\lambda+\alpha\lambda\ \frac{\Gamma^2}{h_0^2}\right)\cos\phi. \qquad (12)$$

From Eq. (4) for $\dot{m}_y(t)$ also one arrives at the same set of Eqs. (11) and (12). This is as it should be since the dynamics is invariant with respect to rotation in the x-y plane.

Now define $\alpha\cos\phi=X$ and $\alpha\sin\phi=Y$. It should be noted that $\alpha\sin\phi$ is directly proportional to the hysteresis loss per cycle, as defined in Eq. (6). Rewriting Eqs. (11) and (12) in terms of X and Y, one gets

$$2(h+\omega)X+\lambda Y=2\Gamma m_0, \qquad (13)$$

and

$$\lambda\left(1+\frac{\Gamma^2}{h_0^2}\right)X-2(h+\omega)Y=-\lambda\ \frac{\Gamma(h+\omega)}{h_0^2}m_0$$
$$+2\lambda\ \frac{\Gamma}{h_0}\tanh\beta h_0. \qquad (14)$$

Treating X and Y as unknown variables, for which Eqs. (13) and (14) are the determining linear equations, one immediately arrives at the following:

$$X=\frac{1}{D}\left[4(h+\omega)\Gamma m_0-\lambda^2\ \frac{\Gamma(h+\omega)}{h_0^2}m_0+2\lambda^2\ \frac{\Gamma}{h_0}\tanh\beta h_0\right] \qquad (15)$$

and

$$Y=\frac{1}{D}\left[2\lambda\ \frac{\Gamma m_0}{h_0^2}(h+\omega)^2+2\lambda\Gamma m_0\left(1+\frac{\Gamma^2}{h_0^2}\right)\right.$$
$$\left.-4\lambda\ \frac{\Gamma}{h_0}(h+\omega)\tanh\beta h_0\right], \qquad (16)$$

where

$$D=4(h+\omega)^2+\lambda^2\left(1+\frac{\Gamma^2}{h_0^2}\right).$$

Using the above expressions for X and Y in Eq. (10), a somewhat lengthy algebra leads to the following equation:

$$m_0=\frac{h+\omega}{h_0}\tanh\beta h_0. \qquad (17)$$

As in Ref. [15] we take $h=m_0$ and then Eq. (17) is a transcendental equation that determines the fixed value of the z component of the magnetization in the attractor and has to be determined numerically. Once that is done, one can find out the values of X and Y and thus the values of α and ϕ using Eqs. (15) and (16). However, it turns out that Y vanishes identically always. Since the denominator D in the expression (16) is positive definite, it is enough to show that the numerator is zero. But that follows simply by using Eq. (17) to evaluate the numerator.

We have checked the correctness of the analytical results presented here by numerically integrating the set of Eqs. (3), (4), and (5) for m_μ ($\mu = x, y, z$), starting from the initial condition $m_z = 1$, $m_x = m_y = 0$. We find that the solution always converges to an attractor of the type described in the paper. In each case, the constant value of m_z to which the solution converges turns out to be the same as given by the solution of the transcendental Eq. (17), which we solve separately for the same set of parameter values. The time axis is divided into intervals of length $T = \pi/\omega$ and for every such interval, we compute A, the hysteresis loss for that cycle, as defined in Eq. (6). We find that A invariably converges to zero, no matter what the values of the parameters of integration (Γ, ω, λ, and β) are. Physically speaking this implies that the projection of the magnetization vector on to the x-y plane eventually becomes parallel to the instantaneous external field vector and then stays that way. We infer that the nonvanishing values of hysteresis loss reported in Ref. [5] are only finite time effects. In summary, we utilize symmetry arguments and numerical investigation of the equations (3), (4), and (5) to infer that the z component of magnetization approaches a constant value asymptotically. Using this knowledge as well as the forms of $m_x(t)$ and $m_y(t)$ that we get from symmetry arguments, we derive the analytical solution for the asymptotic periodic attractor. This solution does not exhibit any hysteresis loss.

ACKNOWLEDGMENT

D.B. acknowledges the University Grants Commission of India for financial support.

[1] G. S. Agarwal and S. R. Shenoy, Phys. Rev. A **23**, 2719 (1981); S. R. Shenoy and G. S. Agarwal, *ibid*. **29**, 1315 (1984).
[2] M. Rao, H. R. Krishnamurthy, and R. Pandit, J. Phys. Cond. Matter Lett. **1**, 9061 (1989); Phys. Rev. B **42**, 856 (1990).
[3] M. Rao and R. Pandit, Phys. Rev. B **43**, 3373 (1991).
[4] D. Dhar and P. B. Thomas, J. Phys. A **25**, 4967 (1992); P. B. Thomas and D. Dhar, *ibid*. **26**, 3973 (1993).
[5] M. Acharyya, B. K. Chakrabarti, and R. B. Stinchcombe, J. Phys. A **27**, 1533 (1994).
[6] P. Jung, G. Gray, R. Roy, and P. Mandel, Phys. Rev. Lett. **65**, 1873 (1990).
[7] A. Hohl, H. van der Linden, R. Roy, and G. H. Goldsztein, Phys. Rev. Lett. **74**, 2220 (1995).
[8] W. S. Lo and R. A. Pelcovits, Phys. Rev. A **42**, 3973 (1990).
[9] W. Wu, B. Ellma, T. F. Rosenbaum, G. Appeli, and D. H. Reich, Phys. Rev. Lett. **67**, 2076 (1991).
[10] T. Tome and M. J. de Oliveira, Phys. Rev. A **41**, 4251 (1990).
[11] T. Zhou, F. Moss, and P. Jung, Phys. Rev. A **42**, 3161 (1990).
[12] A. Simon and A. Libchaber, Phys. Rev. Lett. **68**, 3375 (1992).
[13] M. C. Mahato and S. R. Shenoy, Phys. Rev. E **50**, 2503 (1994).
[14] V. Banerjee and S. Dattagupta, Phys. Rev. B **50**, 9942 (1994).
[15] V. Banerjee, S. Dattagupta, and P. Sen, Phys. Rev. E **52**, 1436 (1995).